Toward a Topological Treatment of the Non-strictly Ordered 2×2 Games

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Abstract

The 2×2 games, the the simplest of all games, serve as the workhorses of applied game theory. Robinson and Goforth[10] introduced a systematic, topologically-based treatment of the relations among the 144 2x2 strictly ordinal games, but there is no systematic general treatment that includes games with ties. This paper introduces a systematic topologically-based approach to the the entire set of 1413 ordinal games. The nonstrictly ordinal 2×2 games are located relative to the strict-ordinal games, some properties of the topological space are described and a small sampling of observations based on the method is provided.

1 Introduction to the Nonstrict Games

2×2 games are characterized by 8 numbers, and the space of 2×2 games is therefore an infinite 8-space. The most familiar classification divides the space by restricting attention to games which differ ordinally. Relations among the 144 strictly ordinal 2×2 games have been presented systematically in Robinson and Goforth [10] using a preference-based topology.[1] This paper introduces a topological approach to the the set of non-strictly ordinal 2×2 games, which is to say ordinal games including those with ties.

Games with ties are very often used as illustrations and in experimental studies. Lacking in most treatments is a rigorous notion of what games are similar to the example. Without a notion of similarity - a topology - results are little more than anecdotes. It is necessary to examine the neighbourhood of a game to check the robustness of solutions in the face of perturbations in the payoff structure. Small perturbations may change the equilibrium payoffs or strategies. Results may be sensitive to some perturbations and not others. Even when an equilibrium payoff changes, equilibrium strategies may not.[2]

After a brief review of the literature in section 2, we introduce the topological approach for strictly ordinal games in section 3. In section 4 we extend the approach to allow for ties. Section 5 provides a classification of games with ties and describes the relationships among the subspaces.

2 Literature

Rapoport and Guyer [8] published *A Taxonomy of the 2x2 Games* in 1966, following it in 1967 with *The 2x2 Game*[9] a book written with D.G. Gordon which extended the results and reporting a considerable amount of empirical research. Their work was taxonomic, which is to say they proposed a set of nested categories largely based on solution concepts. If there was a dominant strategy, for example, the games belonged to one class; if not they belonged to another. They offered the now-standard observation that there are 78 strategically distinct strictly ordinal 2×2 games. [3] Barany, Lee and Shubik [1] classified the 78 distinct 2x2 games using only 24 distinct graphs on four by four grids. However, they did not take into consideration the players' individual strategies and did not consider the structure of the game important.

In 1986 Fraser and Kilgour [2, 3] extended the taxonomic approach to nonstrict games, following the analysis with *A Taxonomy of All Ordinal 2x2 Games*[4] in 1988. They also developed a new system for numbering the games based on patterns in the payoffs rather than the supposed behaviour of players. Using the counting rules proposed by Rapoport and Guyer, they concluded there were 726 ordinal games when games with one or more ties are included, and used a computer program to classify them.

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[1] Rapoport and Guyer[8] provided an earlier and much less rigorous partial treatment.

[2] See the final chapter of [10] for an example of a systematic sensitivity analysis of a familiar result.

[3] There are 576 ways to create strictly ordinal bimatrices. Since swapping rows or columns should not affect the game the number can be reduced by a factor of four. If players are also interchangeable half of the non-symmetric games can also be discarded. Interchangeability is a strong assumption and it has deleterious effects on any attempt to identify relationships among the games. For a fuller discussion see *The Topology of the 2x2 Games: a new periodic table*.

In 1990 Kilgour and Fishburn [5] extended the analysis to allow for intransitive preferences. Transitivity requires that if $a > b > c$ then $a > c$. If we allow $a > b > c > a$ we have cyclic preferences. In most economic analysis cycles are ruled out, but it can be argued that a reasonable person might still exhibit cyclic preferences. Whether the argument is convincing or not it introduced an additional variety of games to consider. Robinson and Goforth introduced a different approach in a series of conference papers beginning in 2001. Beginning with the question 'what does it mean for games to be near each other?' they go on to map the relationships among games in a topological space based on the preference relation[10]. This paper extends their approach to the non-strictly ordinal games.

3 The topology of the 2×2 games

Before extending the topological treatment of the strictly ordinal games to games with ties we set the scene with a brief introduction to the topological space of the 2×2 games. The presentation is based on Robinson and Goforth [10]

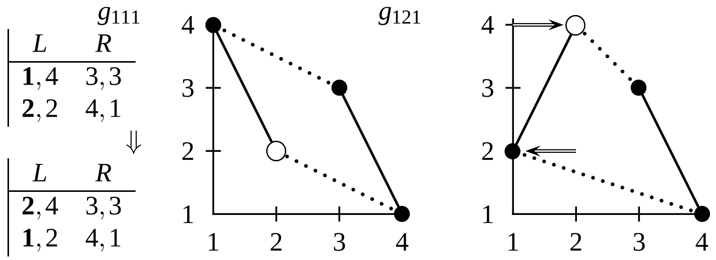
Every 2×2 game is related to every other in the sense that some transformation converts the payoff structure for one into the payoff structure for the other. A complete graph, showing all 10226 transformations among the 144 games edges would be easy to create but essentially useless. To define meaningful neighbourhoods, the key step in defining a topological space, we need to characterize the smallest significant change in the payoff matrix.

Happily, preferences as we think of them in economics provide enough structure to induce a topology on the ordinal 2×2 games. Obviously a change affecting the payoffs of one player is smaller than a change affecting two players. The closest games are therefore those games that differ only in that a small change has been made in the sequence of the four numbers describing the ordinal payoffs for one player. The topology thus induced is rooted in the structure of *preferences*.

3.1 The swap operator for strictly ordinal games

Let the payoff least preferred by the row player R be designated with ordinal 1 and the most preferred with a 4.[4] If the least preferred outcome becomes more and more attractive to R, it will eventually be preferred to the outcome with ordinal 2. When this switch in preference occurs, the effect on the payoff matrix is to exchange the positions of the 1 and 2 in the matrix for the Row player. $R_{12} = \{1 \rightarrow 2\ AND\ 2 \rightarrow 1\}\ \ for\ Row$ This is an example of a minimal transformation of one strictly ordinal game to another.

Changes like this in the payoffs might result from *small* changes in information, preferences, or technology, or might result from small errors in identifying games. A player might, for example, receive a very small amount of new information. She might then reconsider the outcome she had originally ranked 1, and decide that it is slightly better than she first thought, and that it is superior to the outcome she had previously ranked 2. She would naturally relabel the two outcomes, resulting in a different payoff matrix, and hence a different game. The new game is close to the old game in that it is reached as a result of a small perturbation in one player's information set. The game is also close in the sense that it might be mistaken for the original game or it might evolve into the other as a result of a small exogenous change in the underlying technical conditions.[5]



L	R
1,4	3,3
2,2	4,1

L	R
2,4	3,3
1,2	4,1

Figure 1: Example: The swap R_{12} exchanges the ordinal values for the two lowest-ranked outcomes for the row player

Figure 1 illustrates $R_{1,2}$, a '1 for 2' swap for the row player in the Prisoner's

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[4] Defining the first ordinal as the one corresponding to the lowest real payoff provides the most convenient basis for graphing games in ordinal payoff space.

[5] Notice that swaps are independent of the payoff matrix structure and applied to the ordinal payoffs wherever they appear in the payoff matrix.

Dilemma.[6]. There are six such minimal exchanges:

$$\begin{aligned} \boldsymbol{S} &= \{X_{ij} | i \in \{1..3\}, j = i+1, X \in \{R,C\}\} \\ &= \{R_{12}, R_{23}, R_{34}, C_{12}, C_{23}, C_{34}\} \end{aligned}$$

where S_{ij} changes the rank of the outcome originally ranked i by the player S to rank j and the rank of the outcome originally ranked j to rank i. The six row and column swaps applied to a given strictly ordinal game generate the six nearest strictly ordinal neighbours.

A combined swap operation in which the same swap is made for the row and the column players is a symmetric operation, S_{ij}. A symmetric operation preserves symmetry if symmetry is present initially. Under symmetric operations, the symmetric games form a subspace. In addition, symmetric operations beginning with non-symmetric games generate other subspaces of interest.

3.2 the periodic table

Repeated applications of the swap operators will generate a graph that includes every conceivable 2×2 strict ordinal game. The graph displays useful periodicity and symmetries that allow the games to be arrayed systematically. Robinson and Goforth call the resulting table "the periodic table of the 2×2 games" and show that it can be used to predict the properties of games and groups of games.

Figures 6 to 9 show a very convenient arrangement of order graphs for the 144 games produced this way/footnote. There are four "layers" of 36 games. Layers are generated using only $C_{12}, C_{23}, R_{12}, R_{23}$. The games in a layer have one common feature: $C_{12}, C_{23}, R_{12}, R_{23}$ leave the positions of the most preferred elements unchanged for both players.

Each of the layers is a subspace of the larger space of the 2×2 games. Layers can be pictured as 6x6 checkerboards. Each square represents an equivalence set in the space of reals and is represented by a particular 2×2 strictly ordinal game. $C_{12}, C_{23}, R_{12}, R_{23}$ swaps carry us across the boundary to a game above, below, left or right of the first. These games are the nearest neighbours. Although the C_{34}, R_{34} swaps are not shown on the checkerboard, the image is still helpful for visualizing the relationships among the games. The C_{34}, R_{34} swaps connect the four layers.

Each layer, it turns out, can be graphed as a torus[7] as in figure 3, which shows a mesh with 36 intersections first rolled into a tube then bent to form a donut. These toroidal layer structures are the largest simple topological features of the 2×2 games.

Layer 3 provides an example of how the periodic table captures the topological basis of a familiar category. In the Rapoport, Guyer and Gordon typology the 36 games which have the payoff combination $(4,4)$ constitute a "phylum" that the authors call the "no conflict games"[8, 9]. The games all appear on layer three, which which is a compact set of games in the topological space induces by the swap operator.

3.3 boundaries between games

The sets of 8-vectors represented by a given strictly ordinal 2×2 gamesare open sets. The edges that form the boundary between a game and any neighbour are defined by the presence of at least one tie for either player. In the checkerboard representation the lines represent games with ties. Where horizontal and vertical boundaries meet there are two ties, one for the row player and one for the column player.

We can imagine extracting just the edges as a separate object. That object can be understood as a sponge, a membrane enclosing bubbles of space of a higher dimension. For a given square on the checkerboard there are only four sides and four corners because the C_{34}, R_{34} swaps are suppressed. Figure 2 shows how the checkerboard example on a layer maps onto a volume with facets forming the boundary with each of the six neighbours.

4 Introducing ties formally: the half-swap

A *half-swap* brings a pair of consecutive ordinal payoffs to equivalence. For example $r_{12} = \{1 \rightarrow 2 \ OR \ 2 \rightarrow 1\}$, the half-swap version of R12, makes the two lowest payoffs for a player equal. Applied to a strictly ordinal game, it yields a game on the boundary between that game and one of its strictly ordinal neigh-

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[6] The effect of the change in the matrix is shown on the right using "order graphs" in payoff-space. In an order graph, Row's payoffs increase to the right and column's to the top. The ordinal payoff pairs in each column of the payoff matrix are connected with a solid line and the pairs in the rows are connected with a dotted line. (The different lines correspond to "inducement correspondences for the Nash situation," as defined by Greenberg [7]) With the solid and dotted lines, the figures on the right of Figure 1 are complete strategic form representations of the 2×2 games presented in matrix form on the left. When the payoff space is ordinal, as in this case, we call the graph an 'order graph'.

[7] Orthogonal cyclic operations of order three or more always generate a torus.

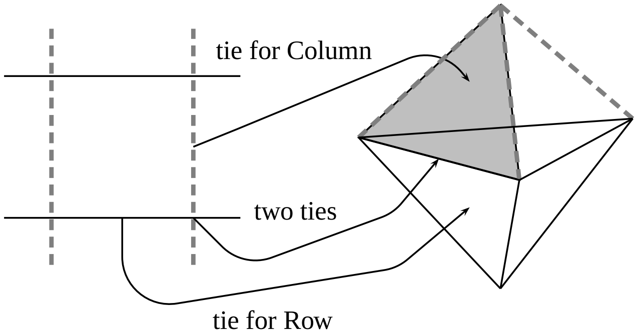


Figure 2: the skin of a typical game

bours. The new game has only three distinct outcomes for Row. The two games in figure 4 illustrate a r_{12} half-swap.

Kilgour and Fraser[4] introduced a convenient approach to the 2×2 games with the observation that a player's payoffs fall into one of eight preference orderings. They began with the degenerate case in which all payoffs are equal and added variety to generate eight distinct payoff patterns:

A	1,1,1,1	E	1,2,2,2
B	1,1,1,2	F	1,2,2,3
C	1,1,2,2	G	1,2,3,3
D	1,1,2,3	H	1,2,3,4

Notice that each player will have an ordering and that, except for ordering A, each ordering corresponds to many matrices. Any 2×2 can be classified with two letters. For example,

	L	R	L	R
U	3.0, 5.0	2.0, 5.0	2, 2	1, 2
D	7.0, 5.0	7.0, 4.0	3, 2	3, 1

is in category GE (ie (1,2,3,3),(1,2,2,2)), with the row pattern identified first.

With $64 = 8 \times 8$ possible combinations, this construction leads to a count of 1413 2×2 ordinal games instead of the conventional figure of 726[6, 2, 3, 4].

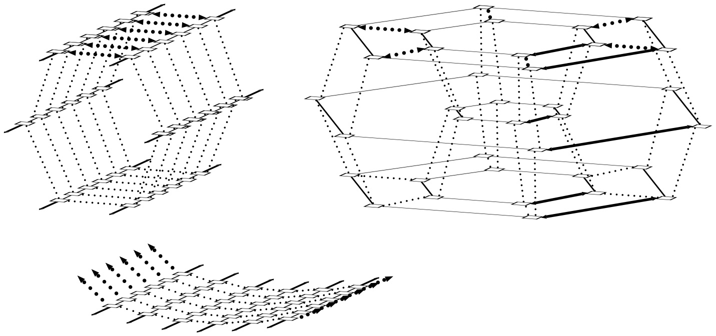
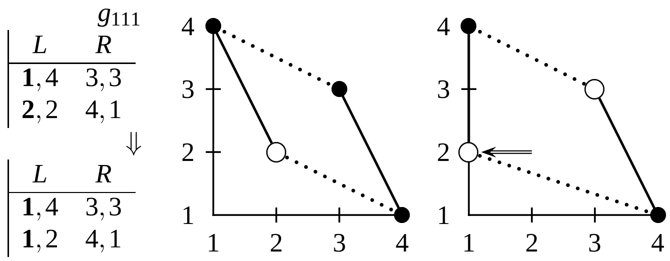


Figure 3: Torus



L	R
1, 4	3, 3
2, 2	4, 1

L	R
1, 4	3, 3
1, 2	4, 1

Figure 4: Example: the lower half-swap r_{12}

Kilgour and Fraser's notation does not capture information about which games are adjacent. Any game with pattern 1,1,2,3 (D above) is a half swap from two games with 1,2,3,4. An alternative - and admittedly clumsy - notation, the player's payoffs are defined as 12,3,4 and deployed in a bi-matrix as follows.[8] The game with the repeated payoff for the row player is shown between the two strict ordinal games.

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[8] The intuition of this notation is that ties occupy multiple positions in a ranking. If two are tied for top spot, the next rank is third, not second.

Kilgour and Fraser				H	G	F	D	E	C	B	A
		Distinct		4		3			2		1
		Payoffs		1,2,3,4	1,2,34	1,23,4	12,3,4	1,234	12,34	123,4	1234
A	1	1234	●●●●	6	3	3	3	1	3	1	**1**
B		123,4	●●● ●	24	12	12	12	4	6	**4**	1
C	2	12,34	●● ●●	36	18	18	18	6	**12**	6	3
E		1,234	● ●●●	24	12	12	12	**4**	6	4	1
D		12,3,4	●● ● ●	72	36	36	**36**	12	18	12	3
F	3	1,23,4	● ●● ●	72	36	**36**	36	12	18	12	3
G		1,2,34	● ● ●●	72	**36**	36	36	12	18	12	3
H	4	1,2,3,4	● ● ● ●	**144**	72	72	72	24	36	24	6

Table 1: Categorization of the 1413 ordinal 2×2 games

$$\begin{array}{c|cc} & L & R \\ \hline U & \mathbf{1}, 4 & 3, 3 \\ D & \mathbf{2}, 2 & 4, 1 \end{array} \xrightarrow{r12} \begin{array}{c|cc} & L & R \\ \hline U & \mathbf{12}, 4 & 3, 3 \\ D & \mathbf{12}, 2 & 4, 1 \end{array} \xleftarrow{r12} \begin{array}{c|cc} & L & R \\ \hline U & \mathbf{2}, 4 & 3, 3 \\ D & \mathbf{1}, 2 & 4, 1 \end{array}$$

5 Classifying nonstrict games

Table 1 enumerates the games by number and type of tie.[9]

5.1 Relationship of Table 1 to the periodic table.

The periodic table is an arrangement of the 144 strictly ordinal 2×2 games. Table 1 is an arrangement of 1413 ordinal games. In Table 1 we imagine that moving to the right represents applying a column half-swap to any of the group of games in the section to the left. The bottom row of Table 1 has four cells. The first cell on the left contains the 144 strictly ordinal games. The next cell contains the result of applying the three column half-swaps once. Recalling that each swap connects two games, there are 72 boundaries associated with each swap or half swap.[10] The second cell therefore contains three sets of 72 in which the column player has one tie (Kilgour and Fraser's HG, HF, and HD). The three sets in which the row player has one tie are enumerated in the cell above the number 144. These $6 \times 72 = 432$ games include the games represented by the lines of the checkerboard in subsection 3.3.

The points at which lines cross on the checkerboard appear as the $9 \times 36 = 324$ games enumerated in the block above and to the right of the 144 strictly ordinal games. That cell has nine groups of thirty-six, each group representing one of the nine ways to combine a row and a column half-swap. and each b in halfing the number of games again. All 900 games in the lower left four blocks of Figure 1 are represented as areas, lines or points in the periodic table. The rest of Table 1 is not represented in periodic table.

5.2 Games far from the periodic table

The top row (A*)and right column (*A) of Table 1 includes all games in which at least one player is completely indifferent. Block AA is the unique completely degenerate game in which both players have choices but neither care about outcomes. It might be argued that all the A games are degenerate. A player that is indifferent has no self-interest at stake and therefore the game-theorist's usual first-order approximation of behaviour is unavailable. On the other hand, we simply do not know how people behave in such situations. Envy, altruism, or a sense of fairness might come into play, and in fact such games, or games that are very close to them, might be ideally suited for exploring how these motives enter behaviour.

Row and column C are also of special interest because players have two ties that each appear in two outcomes. In one third of these games a players is completely in control of her payoff, the strongest possible case of a dominant strategy, and if purely self-interested, indifferent to the other player's choices. Like the "A" games, these might be considered degenerate. In another third of the game the player is completely at the mercy of the other player. The C payoff patterns have an elemental simplicity. As a result, block CC contains what might be seen as the archetypes of the 2×2 games. The most common examples of coordination games and constant sum games are at the exact center of this block.[11]

Block CC stands out in another way. The numbers on the positive diagonal of Table 1 are perfect squares, with one exception. The set (CC) based on

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[9] To get the count proposed by Guyer and Hamburger or Kilgour and Fraser, expand the groups of games on the diagonal of Table 1, then simply delete all the games above (or below) the extended main diagonal. (Notice for example that for the 144 games in the lower left, the extended main diagonal consists of the 12 symmetric games.)

[10] Each boundary is shared by two games.

[11] This is a statement that only makes sens if you have a topology on the games.

r12,r34,c12,c34 half swaps contains only 12 games. The irregularity stems from a foundational assumption that is explored in depth in Robinson and Goforth (2005). The CC set also contains the only symmetric 2 × 2 game that contains no outcome with equal payoffs:

	L	R
U	12,34	34,12
D	34,12	12,34

or, more familiarly,

	L	R
U	-1,1	1,-1
D	1,-1	-1,1

Figure 5 is a directed graph representing the all possible ways to collapse the strictly ordinal payoffs for *one* player. It represents a movement from bottom to top in Table 1. Parallel sides in the figure represent the same half-swap. From the strict ordering, H, there are three possible half-swaps to patterns with three distinct payoffs, D, F, G. From each of these, there are two possible half swaps to patterns of two distinct payoffs, B, C, E, then a final half-swap from each of these to the degenerate case, A.[12]

Table 1 provides an example of how four sets of games collapse as a result of repeated application of half-swaps. The asterisks represent any payoff pattern for the column player.

6 Conclusions

Games with ties are often used in empirical research as well as teaching, and in fact indifference may be very common in practice. We know of no attempt to determine whether ties are common in everyday life or whether people have interesting strategies for dealing with ties. The assumption that players can rank all outcomes strictly is rather strong, as is the assumption that players bother ranking all outcomes. Furthermore, we simply don't know whether the fact that games with ties are a set of measure zero in the 8-space of the 2 × 2 games is relevant in practice. Are people "splitters"who make extremely fine distinctions, or are people generally "lumpers" who consider payoffs equivalent even when they are clearly of different magnitude? Are people consistent within themselves on this matter, and do people behave in a similar way when faced with similar payoffs. These are empirical questions of some importance.

To deal with such questions we do need tools that allow us to think rigorously about which payoff patterns are similar. By extending the topology induced by the

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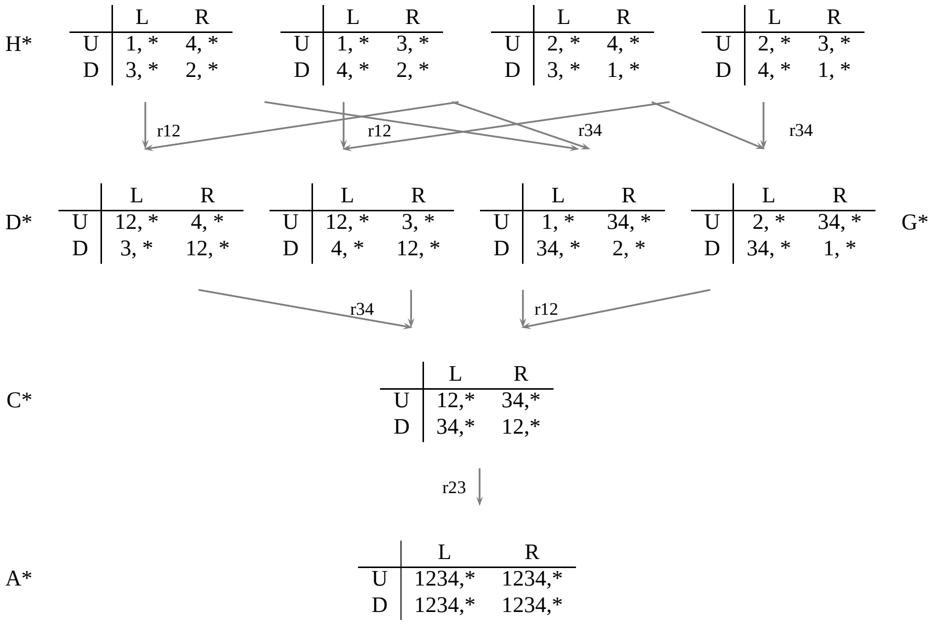
[12] Application of the half swap operations is commutative. r12(r34(H)) = r12(G) = E = r34(F) = r34(r12(H))



H*

	L	R
U	1, *	4, *
D	3, *	2, *

	L	R
U	1, *	3, *
D	4, *	2, *

	L	R
U	2, *	4, *
D	3, *	1, *

	L	R
U	2, *	3, *
D	4, *	1, *

D*

	L	R
U	12, *	4, *
D	3, *	12, *

	L	R
U	12, *	3, *
D	4, *	12, *

	L	R
U	1, *	34, *
D	34, *	2, *

	L	R
U	2, *	34, *
D	34, *	1, *

G*

C*

	L	R
U	12,*	34,*
D	34,*	12,*

A*

	L	R
U	1234,*	1234,*
D	1234,*	1234,*

Table 2: Collapsing a set of games to a smaller set with ties in group C*

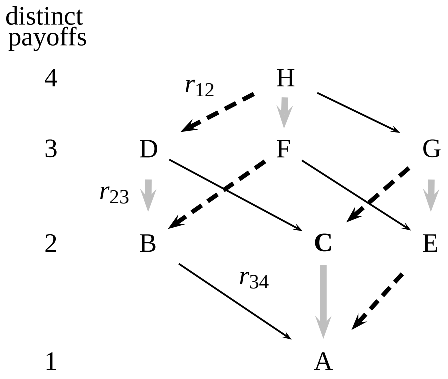


Figure 5: Paths from no ties to all ties

structure of preferences to the non-strictly ordinal 2 × 2 games we have provided some of the necessary machinery.

References

[1] Barany, Imre; J. Lee and Martin Shubik “Classification of Two-Person Ordinal Bimatrix Games” *International Journal of Game Theory* Springer, vol. 21(3), pp 267-90.

[2] Fraser, Niall M. and D. Marc Kilgour. “Nonstrict Ordinal 2x2 Games: A Comprehensive Computer-Assisted Analysis of the 726 possibilities” *Theory and Decision 20* pp 99-121 1986a

[3] Fraser, Niall M. and D. Marc Kilgour. “General Ordinal 2 x 2 Games” *Technical report #172* Dept. of Management Sciences, University of Waterloo 1986b

[4] Fraser, Niall M. and D. Marc Kilgour. “A Taxonomy of all Ordinal 2 x 2 Games” *Theory and Decision 24* pp 99-117. 1988

[5] Fishburn, Peter C. and D. Marc Kilgour. “Binary 2x2 Games” *Theory and Decision 29* pp 165-82. 1990.,

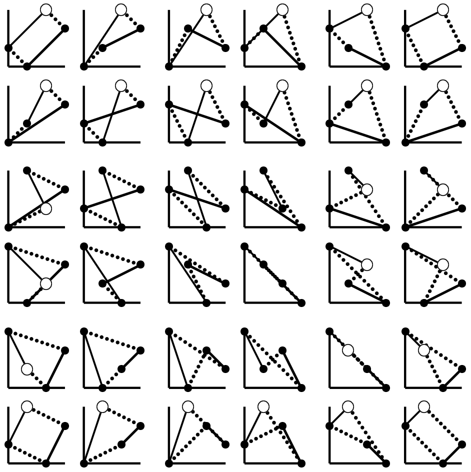


Figure 6: Layer 2

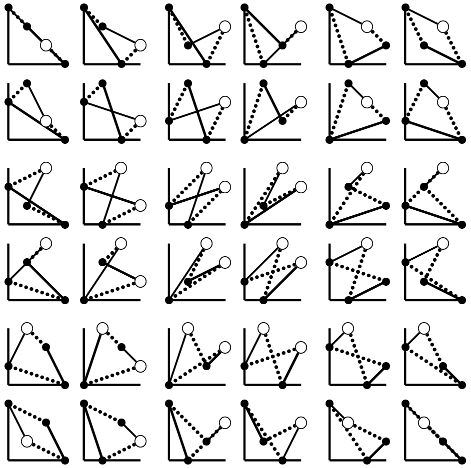


Figure 7: Layer 1

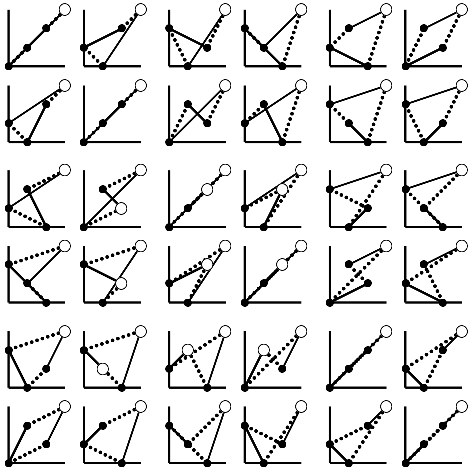


Figure 8: Layer 3

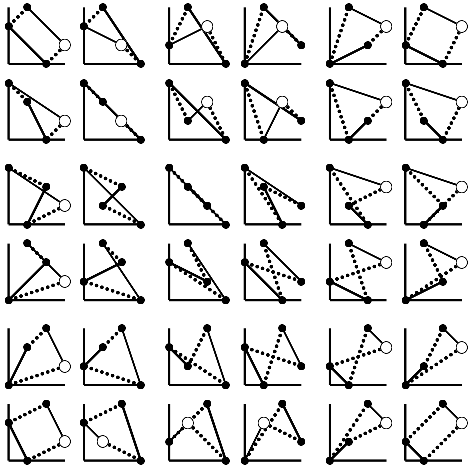


Figure 9: Layer 4

[6] Guyer, Melvin, and H. Hamburger. ‘A Note on “A Taxonomy of 2x2 Games” ’ *General Systems* 13, pp 205-208. 1968.

[7] Greenberg, Joseph. *The Theory of Social Situations: an Alternative Game Theoretic Approach*. Cambridge: Cambridge University Press. 1990.

[8] Rapoport, Anatol, and Melvyn. J. Guyer. “A Taxonomy of 2x2 Games.” *General Systems, vol XXIII*, pp 125-136. 1978.

[9] Rapoport, Anatol, Melvyn. J. Guyer and David G. Gordon. *The 2x2 Game*. Ann Arbour. University of Michigan Press 1976.

[10] Robinson, David and David Goforth. *The Topology of the* 2×2 *games: A New Periodic Table* London. Routledge. 2005.